Boosting Algorithms
Weak and Strong Learning

PAC Learning model

- Some distribution D over domain X
- Examples: $<x, c^*(x)>$
  - c^* is the target function
- Goal:
  - With high probability $(1-\delta)$
  - find h in H such that
  - $error(h,c) < \varepsilon$
- δ and ε are arbitrarily small.

Weak Learning - Confidence

- Assume we are given a learning algorithm
  - with confidence $\delta = ½$
  - but arbitrary small accuracy $\varepsilon > 0$

- Can we boost the confidence?
  - How?!

BoostConfidence Algorithm

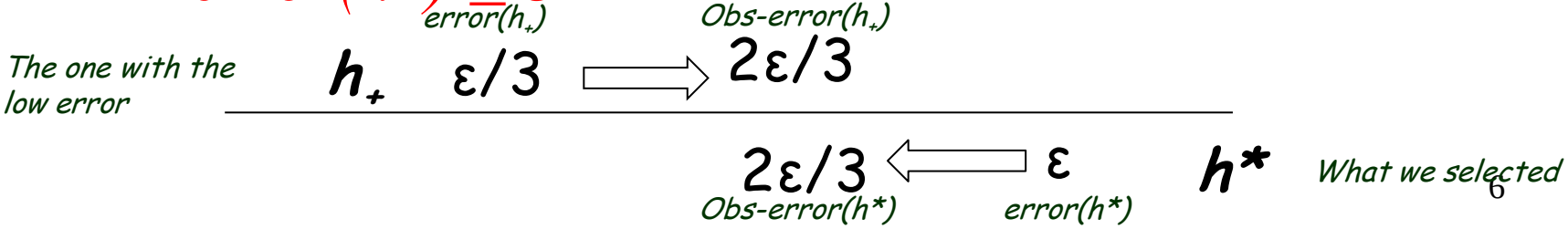
- **Input**: Algorithm **A** and parameter **δ**
- Create $k=log(2/\delta)$ independent problems
  - Sample S_i for i-th copy
  - Run Algorithm **A** on S_i with accuracy $\varepsilon/3$
  - Let h_i be the hypothesis that **A** outputs on S_i
- Take a new sample S of size $m=\varepsilon^{-2} log(2k/\delta)$
  - Return the best hypothesis h_i on S
    - call it h^*

BoostConfidence Analysis S_i

- For each copy i:
  - The probability that $error\ (h_i) < \varepsilon/3$ at least ½
- The probability that
  - some h_i has $error(h_i) < \varepsilon/3$
  - at least $1-2^{-k}=1-\delta/2$
    - Holds for $k=log(2/\delta)$
- Assume this holds!
  - Namely, some h_i has $error(h_i) < \varepsilon/3$
    - denote it by $\boldsymbol{h}_+$

BoostConfidence Analysis *S*

- With probability at least *1- δ/2*
  - for every h_i: $|error(h_i)-[obs\text{-}error(h_i)]| \leq \varepsilon/3$
    - Chernoff bound + union bound using $m = \varepsilon^{-2} \log(2k/\delta)$
- Assume this holds!
- Together, with probability at least *1-δ*
  - $error(h^*) \leq \varepsilon$



Weak Learning - Accuracy

- Assume: $error(h,c) < \frac{1}{2}-\gamma$
- The parameter $\gamma>0$ is small
  - constant
- Intuitively: getting slightly better than chance is easy
- Question:
  Assume **C** is weak learnable,
  is **C** PAC (strong) learnable?

Weak Learning - Definition

Weak Learning

- Algo A weak learns C using H if
- exists **$\gamma > 0$**
- for all c in C
- **for any distribution D**
- for all $\delta > 0$
- Outputs h in H such that
  - with prob 1- δ
  - ***error(c,h) < ½ - γ***

Strong Learning

- Algo A strong learns C using H if
- for all **$\varepsilon > 0$**
- for all c in C
- for any distribution D
- for all $\delta > 0$
- Outputs h in H such that
  - with prob 1- δ
  - ***error(c,h) < ε***

Weak learning – definition

- Why do we need **ANY** distribution
- Example:
  - Consider the following distribution over bits
  - if $x_1=x_2=0$ then $c^*(x)=$ some hard function
  - otherwise $c^*(x)=0$
- Uniform Distribution
  - Predicting random for $x_1=x_2=0$, 0 elsewhere will be correct 87.5%
  - Getting above that is impossible.
- The hard distribution:
  - Sample from the set where $x_1=x_2=0$

Three Weak Learners

- One weak learner
  - only one thing to do !
- Two weak learners
  - what to do if they disagree?
- Three weak learners
  - Can we improve accuracy?

- Example

x	- y	x	- y
11110	- 1	10111	- 0
11110	- 1	01101	- 0
10011	- 1	11011	- 0
01001	- 1	01100	- 0
10001	-1	00000	- 0

Weak learners: single bits

Three weak learners

- First weak learner
  - use the distribution D
  - get h_1
- Second weak learner
  - How can we force new h?
  - Set D_2 s.t.
  - h_1 has error ½
  - Get h_2
    - why $h_2 \neq h_1$?
- Third weak learner
  - What are "interesting inputs"?
  - $h_2(x) \neq h_1(x)$
    - let D_3 be such inputs
  - get h_3
- How will we predict?
  - If $h_2(x) \neq h_1(x)$ using $h_3(x)$
  - Else $h_2(x)$ (or $h_1(x)$)
    - **majority**

3 Weak Learners - Performance

- Define D_2 and D_3
- Dist D_2:
  - Select random b={0,1}
  - If b=0
    - Sample x from D_1 until $h_1(x)=c^*(x)$
  - Else (b=1)
    - Sample x from D_1 until $h_1(x)\neq c^*(x)$
- Dist D_3
  - if $h_1(x)\neq h_2(x)$ sample

- Formally

$$D_2(x)=\begin{cases}\frac{0.5}{1-p}D_1(x) & \text{if } h_1(x)=c^*(x)\\ \frac{0.5}{p}D_1(x) & \text{if } h_1(x)\neq c^*(x)\end{cases}$$

$$p=\Pr[h_1(x)\neq c^*(x)]$$

$$D_3(x)=\begin{cases}\frac{D_1(x)}{Z} & \text{if } h_1(x)\neq h_2(x)\\ 0 & \text{if } h_1(x)=h_2(x)\end{cases}$$

$$Z=\Pr[h_1(x)\neq h_2(x)]$$

3 Weak Learners - Analysis

- Assume, for simplicity, that all WL have error rate $p=\frac{1}{2}-\gamma$

- If all errors are independent, the error of the majority is
  $Error = 3p^2(1-p)+p^3 = 3p^2-2p^3$

- We show next that this holds even if the errors are dependent

3 Weak Learners - Analysis

- Assume all WL are $p=½ -\gamma$

$$D_2(x) = \begin{cases} \frac{0.5}{1-p} D_1(x) & \text{if } h_1(x) = c^*(x) \\ \frac{0.5}{p} D_1(x) & \text{if } h_1(x) \neq c^*(x) \end{cases}$$

$$p = \Pr[h_1(x) \neq c^*(x)]$$

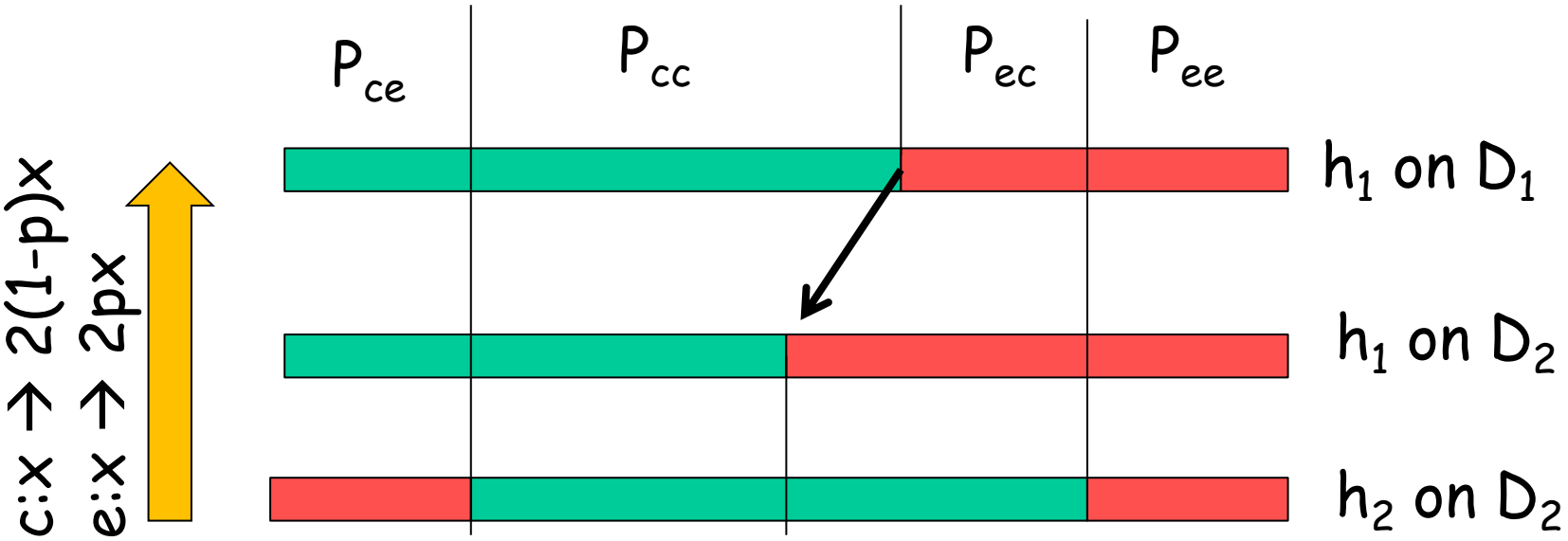
- To convert from a probability x in D2 to D1

  Correct in h1: $x \rightarrow 2(1-p)x$

  Error in h1: $x \rightarrow 2px$

3 Weak Learners - Analysis

- Assume all WL are $p=\frac{1}{2}-\gamma$



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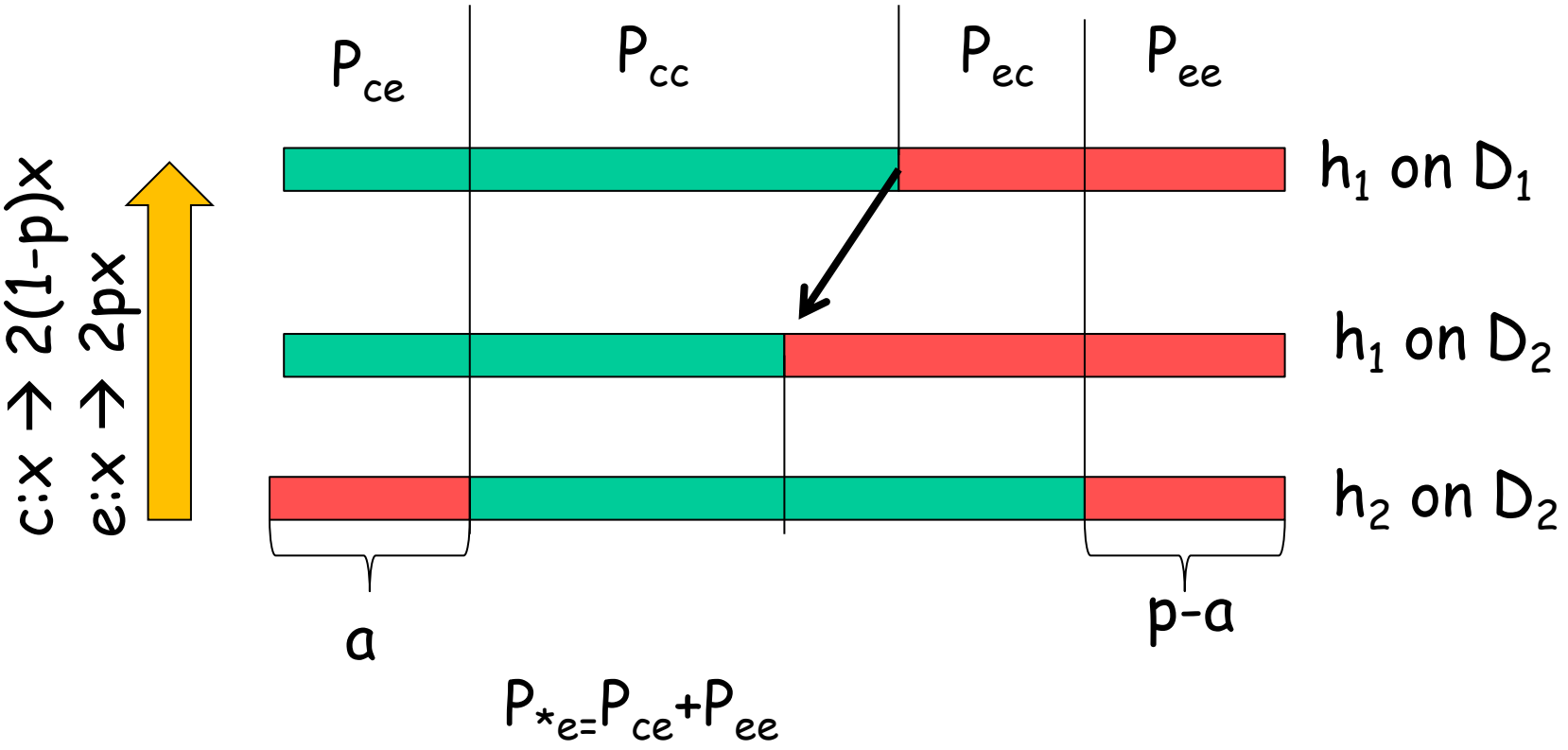

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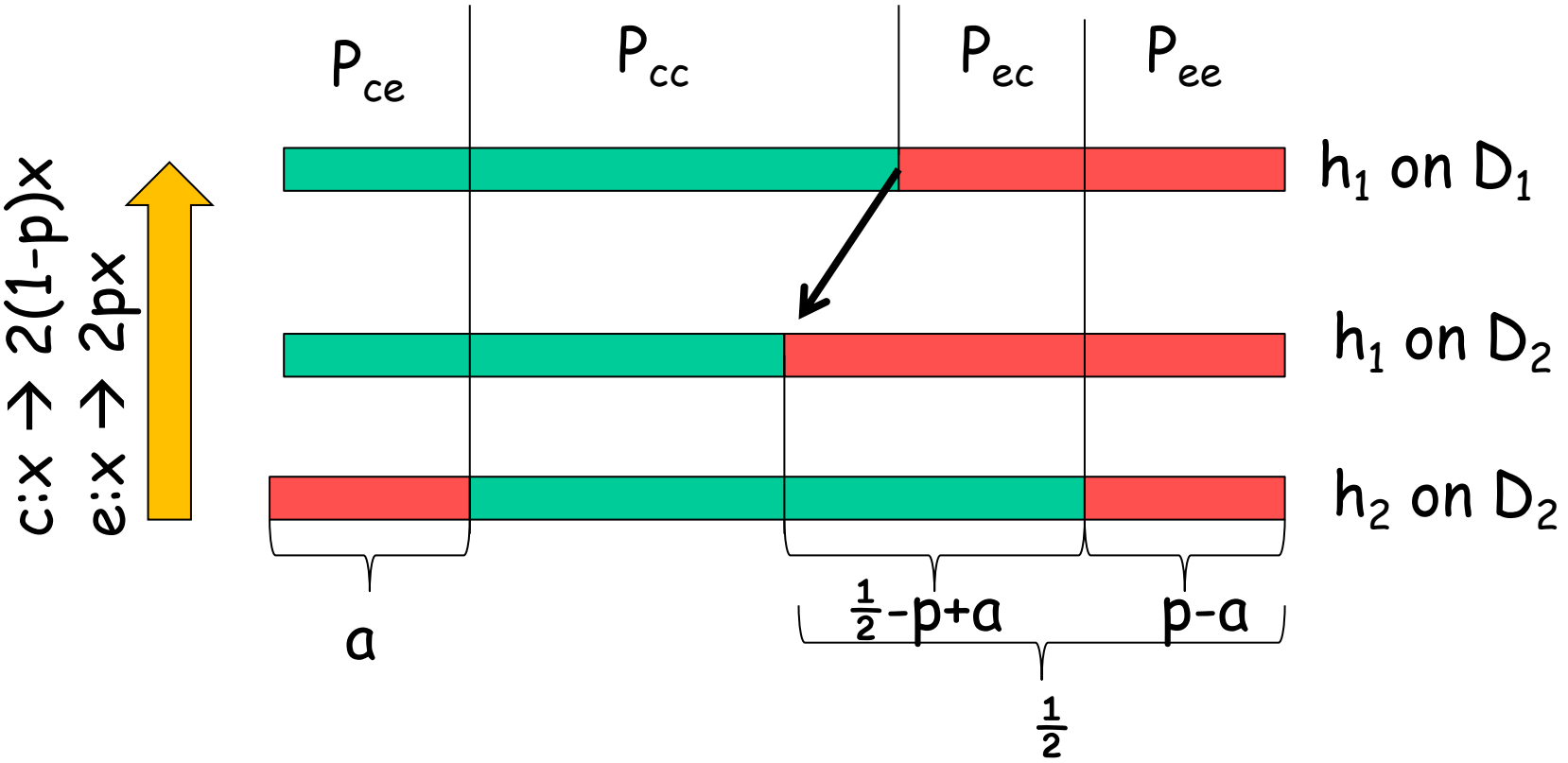

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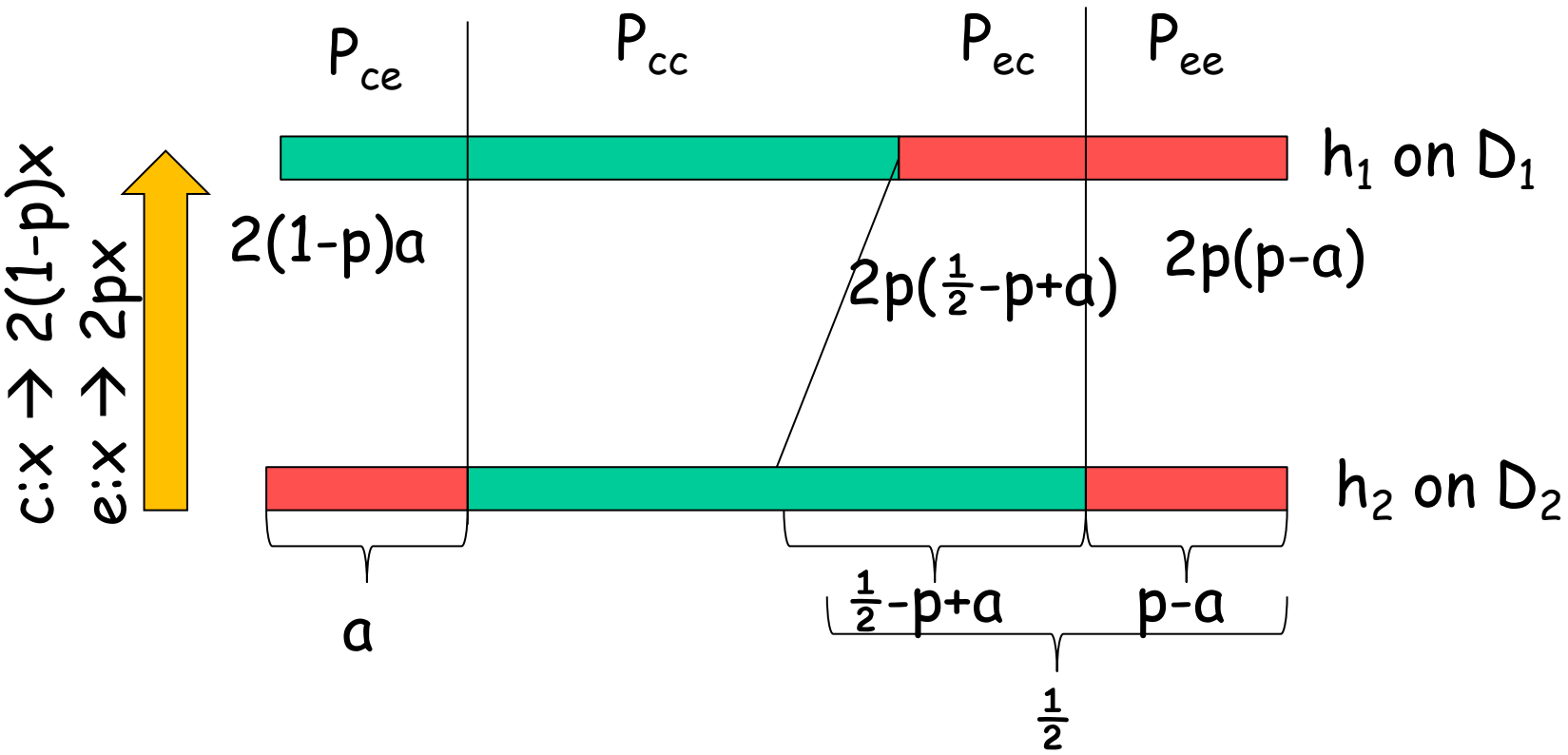

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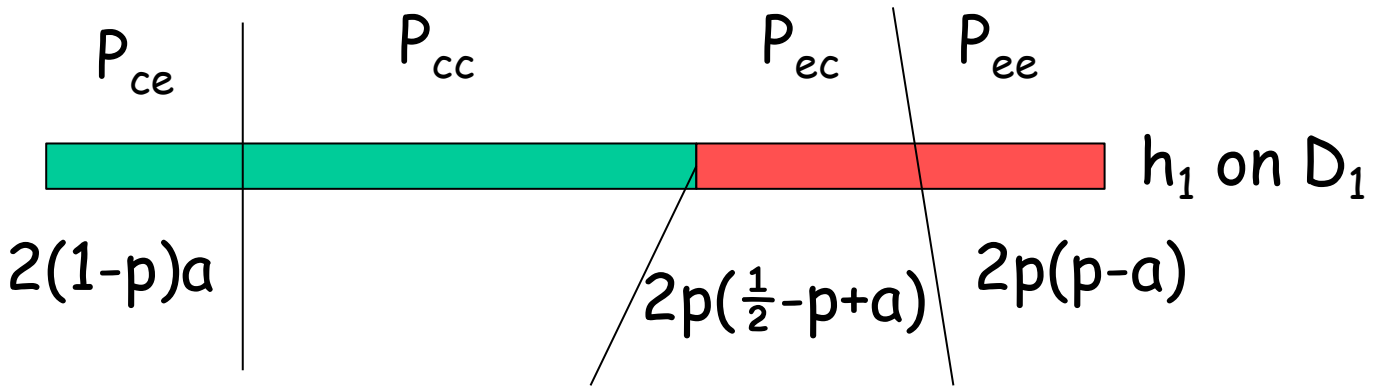
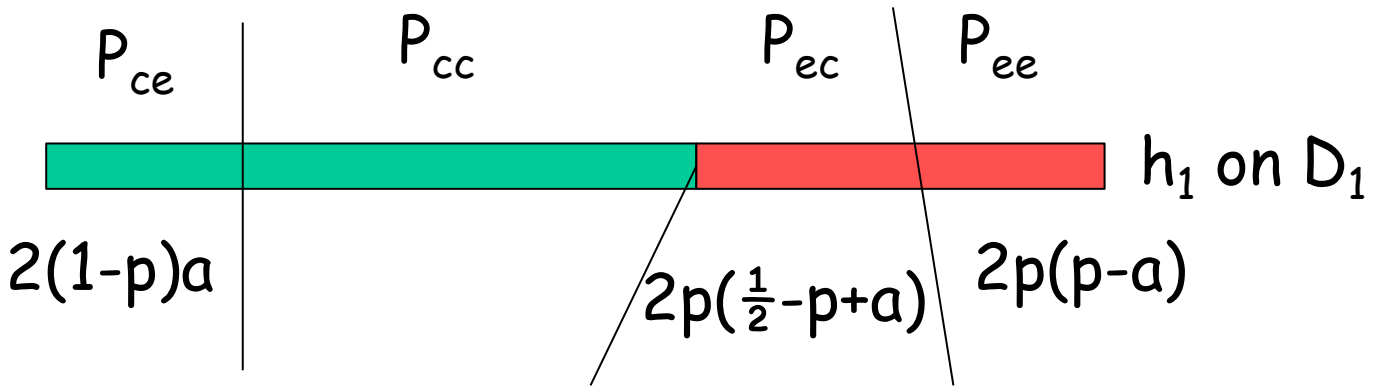


Error = $P_{ee}+p(P_{ec}+P_{ce})=3p^2-2p^3$

3 Weak Learners - Analysis

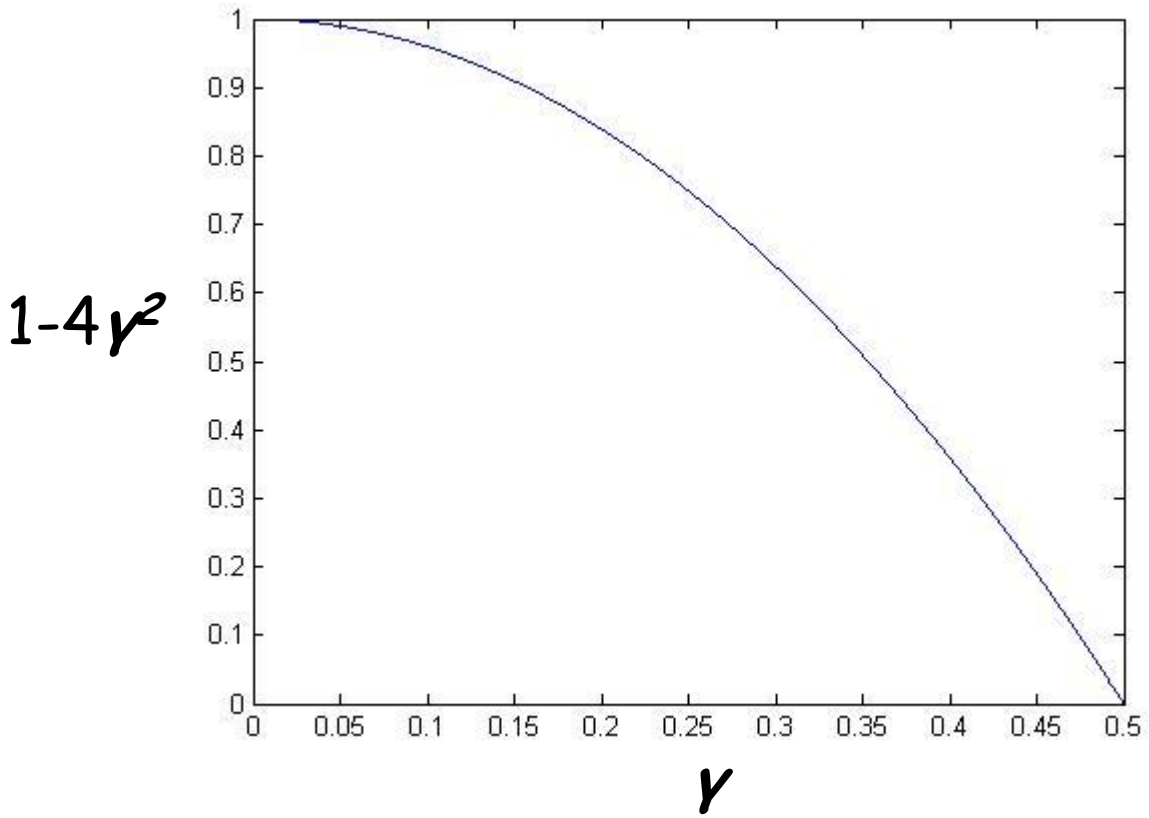
- Assume all WL are $p=\frac{1}{2}-\gamma$



Error = $P_{ee}+p(P_{ec}+P_{ce})=3p^2-2p^3=p(3p-2p^2)=p(1-4\gamma^2)$

3 Weak Learners - Analysis

New Error =Old Error*$(1-4\gamma^2)$



What about more hypothesis

- The CS way
  - Do it recursively
  - Can push down the error arbitrarily
- We show a more constructive way

ADABOOST: ADAPTIVE BOOSTING

AdaBoost: Overview

- Build a linear classifier
  - basic elements, weak learners
  - Ensemble method
- An "online" approach
  - each time add one more classifier
- Fit the sample **S**

- Each time step t
  - have a hypothesis f_t
  - Select a distribution D_t
    - on **S**
  - Find a weak learner h_t
    - w.r.t D_t
  - Add h_t to the hypothesis
    - decide on weight a_t
    - $f_{t+1}(x)=f_t(x)+a_t h_t(x)$
    - predict $\boldsymbol{sign}(f_{t+1}(x))$

AdaBoost: Algorithm

- Weak Learners ***H***

  $h: X \rightarrow \{+1,-1\}$

- Initialization:
  - FIXED Sample
    - $S=\{(x_1,y_1) \ldots,(x_m,y_m)\}$
  - $D_1(x_i) = 1/m$

- Step $t = 1, \ldots, T$
  - Receive h_t
    - WL w.r.t. D_t
  - Define
    - $\varepsilon_t = Pr[h_t(x) \neq c^*(x)]$
    - $\alpha_t = \frac{1}{2} \log (1- \varepsilon_t) / \varepsilon_t$
  - Define D_{t+1},

$$D_{t+1}(x_i) = \frac{D_t(x_i)}{Z_t} \cdot \begin{cases} e^{-\alpha_t} & y_i = h_t(x_i) \\ e^{\alpha_t} & y_i \neq h_t(x_i) \end{cases}$$

$$\propto D_t(x_i) e^{-\alpha_t y_i h_t(x_i)}$$

$$Z_t = \sum_i D_{t+1}(x_i)$$

AdaBoost: Algorithm

- Weak Learners ***H***

  $h: X \rightarrow \{+1,-1\}$

- Initialization:
  - FIXED Sample
    - $S=\{(x_1,y_1) \ldots,(x_m,y_m)\}$
  - $D_1(x_i) = 1/m$

- Prediction

$f(x)=sign(\Sigma \alpha_t h_t(x))$

- Step $t = 1, \ldots, T$
  - Receive h_t
    - WL w.r.t. D_t
  - Define
    - $\varepsilon_t=Pr[h_t(x)\neq c^*(x)]$
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$$\propto D_t(x_i) e^{-\alpha_t y_i h_t(x_i)}$$

$$Z_t = \sum_i D_{t+1}(x_i)$$

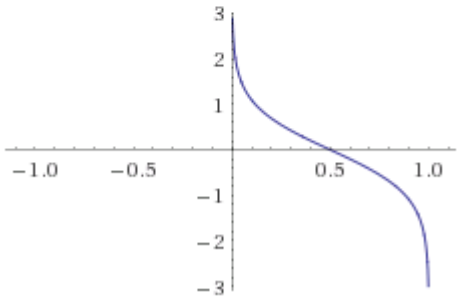
AdaBoost: Intiution

- How do we change the distribution?
  - Error → weight increases
  - Correct → weight decreases
  - Focus on the "hard" examples
- What are the parameters?
  - The weak learning class ***H***
  - The number of iterations ***T***
    - Assume to be inputs

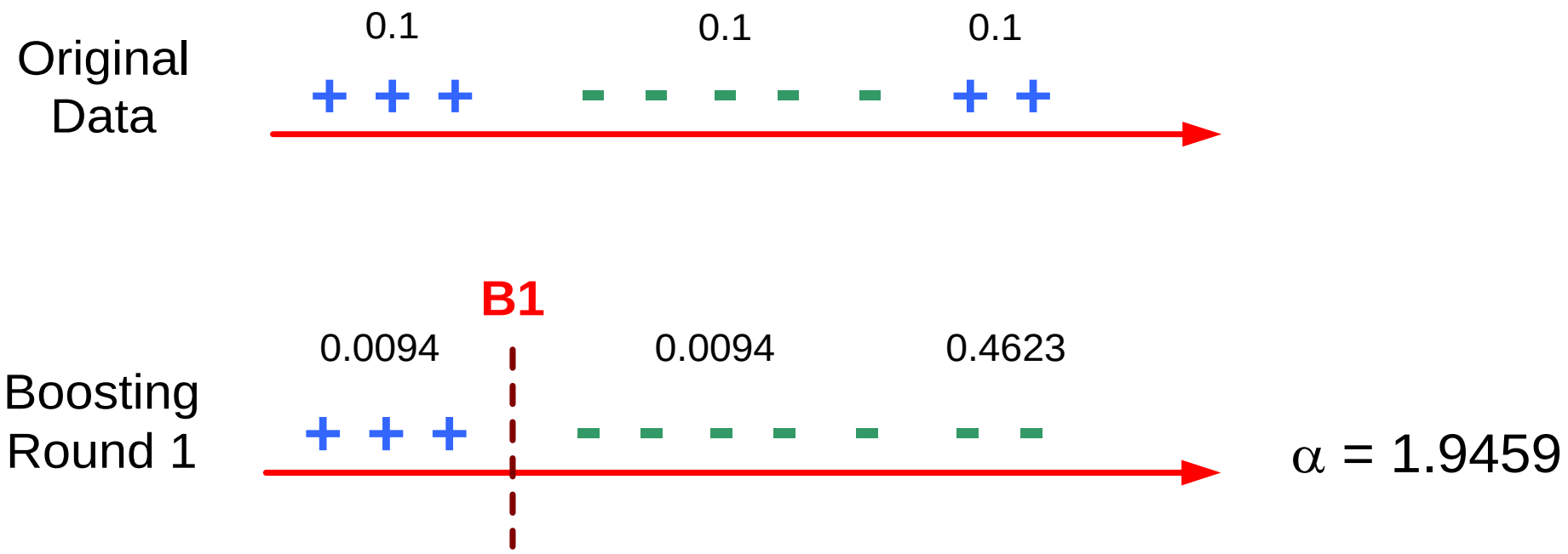
Input:

$$0.5 \log\left(\frac{1-x}{x}\right)$$

Plots:



Illustrating AdaBoost



Slide credit: Evimaria Terzi

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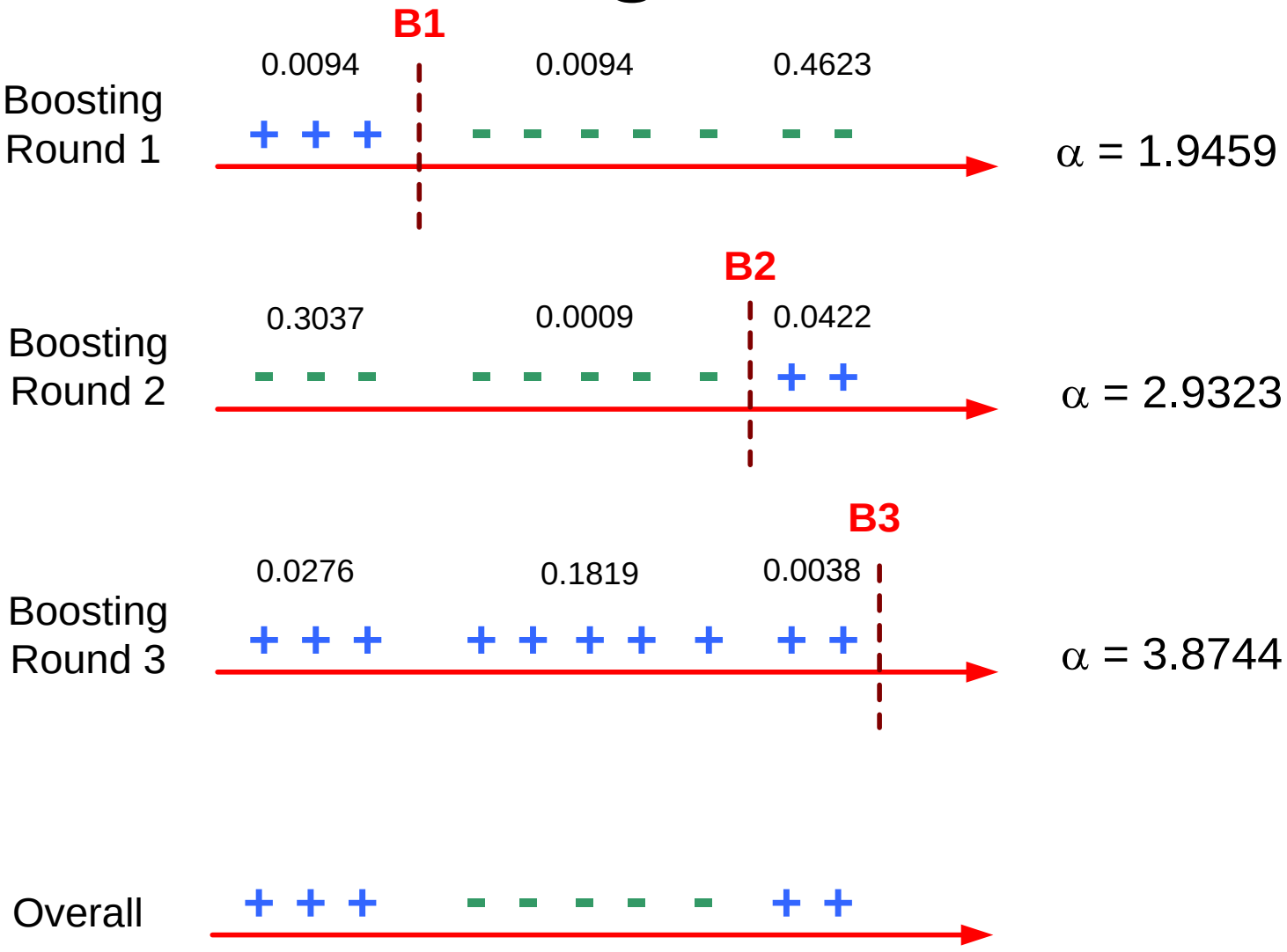

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AdaBoost: Analysis

- Theorem:
  - Given $\varepsilon_1, \ldots, \varepsilon_T$
  - the training error $\boldsymbol{\varepsilon}$ of $\boldsymbol{f}$ is bounded by

$$\varepsilon \leq 2^T \prod_{t=1}^{T} \sqrt{\varepsilon_t (1-\varepsilon_t)}$$

- Proof based on three claims

AdaBoost: Analysis

Claim 1:

$$D_{T+1}(x_i) = \frac{D_1(x_i)e^{-y_i f(x_i)}}{\prod_t Z_t}$$

where $f(x)=\Sigma\alpha_t h_t(x)$

Corollary 1: since $D_1(x_i) = 1/m$

$$e^{-y_i f(x_i)} = mD_{T+1}(x_i)\prod_t Z_t$$

Proof

- For $t+1$ we have

$$D_{t+1}(x_i) = \frac{D_t(x_i)}{Z_t} e^{-\alpha_t y_i h_t(x_i)}$$

- unravel recurrence

$$D_{T+1}(x_i) = D_1(x_i)\prod_{t=1}^{T} \frac{e^{-y_i \alpha_t h_t(x_i)}}{Z_t}$$

$$= D_1(x_i)\frac{e^{-y_i \sum_{t=1}^{T} \alpha_t h_t(x_i)}}{\prod_t Z_t}$$

$$= D_1(x_i)\frac{e^{-y_i f(x_i)}}{\prod_t Z_t}$$

AdaBoost: Analysis

Claim 2:

$$error(f,S) \le \prod_{t=1}^{T} Z_t$$

Proof

$$error(f,S) = \frac{1}{m}\sum_{i=1}^{m} I(y_i \neq f(x_i))$$

$$= \frac{1}{m}\sum_{i=1}^{m} I(y_i f(x_i) \le 0)$$

$$\le \frac{1}{m}\sum_{i=1}^{m} e^{-y_i f(x_i)}$$

z<0 → e^{-z}>1

$$= \frac{1}{m}\sum_{t=1}^{T} m D_{T+1}(x_i) \prod_t Z_t$$

Corollary 1

$$= \left(\prod_t Z_t\right)\sum_{i=1}^{m} D_{T+1}(x_i)$$

$$= \prod_t Z_t$$

AdaBoost: Analysis

Claim 3:

$$Z_t = 2\sqrt{\varepsilon_t(1-\varepsilon_t)}$$

Proof

$$\begin{aligned} Z_t &= \sum_{i=1}^{m} D_t(x_i) e^{-y_i \alpha_t h_t(x_i)} \\ &= \sum_{i:y_i = h_t(x_i)} D_t(x_i) e^{-\alpha_t} + \sum_{i:y_i \neq h_t(x_i)} D_t(x_i) e^{\alpha_t} \\ &= (1-\varepsilon_t) e^{-\alpha_t} + \varepsilon_t e^{\alpha_t} \end{aligned}$$

recall $\sum_{i:y_i \neq h_t(x_i)} D_t(x_i) = \varepsilon_t$

and substitute $a_t = \frac{1}{2} \log (1- \varepsilon_t) / \varepsilon_t$

AdaBoost: Analysis

- Why this value of alpha?
  - $Z_t = (1-\varepsilon_t)e^{-\alpha_t} + \varepsilon_t e^{\alpha_t}$ valid for any α_t
  - Minimize Z_t to reduce error (claim 2)

$$\frac{dZ_t}{d\alpha_t} = -(1-\varepsilon_t)e^{-\alpha_t} + \varepsilon_t e^{\alpha_t}$$

$$e^{2\alpha_t} = \left(\frac{1-\varepsilon_t}{\varepsilon_t}\right)$$

$$\alpha_t = \frac{1}{2}\ln\left(\frac{1-\varepsilon_t}{\varepsilon_t}\right) \Rightarrow Z_t = 2\sqrt{\varepsilon_t(1-\varepsilon_t)}$$

AdaBoost: Analysis

- Theorem:
  - Given $\varepsilon_1, \ldots, \varepsilon_T$
  - the error $\boldsymbol{\varepsilon}$ of $\boldsymbol{f}$ is bounded by $\prod_{t=1}^{T} Z_t$

$$\varepsilon \leq 2^T \prod_{t=1}^{T} \sqrt{\varepsilon_t (1-\varepsilon_t)} = \prod_{t=1}^{T} \sqrt{1-4\gamma_t^2} \leq e^{-2\sum_t \gamma_t^2}$$

$1 + x \leq e^x$

AdaBoost: Analysis

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$1 + x \le e^x$

- For $\gamma_t > \gamma$

$$\varepsilon \le e^{-2T\gamma^2}$$

Additional Issues

- Boosting the margin
- Overfitting
- When to stop
- Advantages
- Weaknesses
- Implementation
- Bagging/ Stacking/ Boosting
- Decision stumps